Ordinary Differential Equations
Exercise Sheet 1

Exercise 1. *Show that Picard's local existence and uniqueness theorem still holds if the continuity assumption of* $\frac{\partial f}{\partial y}$ *in* D *is replaced by continuity in* $D^* := D \setminus \{(t, y_k) : t \in (a, b), k = 1, \dots, n\}$, *assuming in addition that* $\frac{\partial f}{\partial y}$ *is bounded in* D^*.

[Hint: Show that f *is Lipschitz.]*

Exercise 2. *Calculate the first three Picard iterates of the IVPs:*

$$\begin{aligned} y' &= t^2 + y^2, \qquad y(0) = 1, \\ y' &= 1 - y^3, \qquad y(-1) = 3, \\ y' &= t(3 - 2y), \qquad y(1) = 2. \end{aligned}$$

Exercise 3. *Show that the unique solution of the IVP*

$$y' = y \sin(y) + t, \qquad y(0) = \frac{\pi}{2}$$

is defined in all of $\mathbb{R}$.

[Hint: Show that the solution does not become infinite in finite time by deriving an a priori estimate.]

Exercise 4 (Continuous dependence)**.** *Consider the two solutions* $y, z : (a, b) \to \mathbb{R}$ *of the IVPs in a finite interval* (a, b)*:*

$$\begin{cases} y' = f(t, y, \lambda_1), & y(t_0) = y_0 \\ z' = f(t, z, \lambda_2), & z(t_0) = z_0 \end{cases},$$

where $f(t, y, \lambda)$ *is Lipschitz relative to* y, λ *everywhere. Show that for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that:*

$$|z_0 - y_0| + |\lambda_2 - \lambda_1| < \delta \qquad \Longrightarrow \qquad |z(t) - y(t)| < \varepsilon,$$

for every $t \in (a, b)$.

[Hint: Use Gronwall's inequality.]

Exercise 5. *Consider the solutions* y_n *of the sequence of IVPs:*

$$y_n' = \frac{1}{n} y_n + y_n^2, \qquad y(0) = \frac{n-1}{n}.$$

Find if y_n *converges and where.*

Exercise 6 (More general Gronwall's inequality)**.** *Consider the continuous functions* $y, z_2, z_1 : [t_0, T] \to \mathbb{R}$, *with* $z_2 \geq 0$, *satisfying the inequality*

$$y(t) \leq z_1(t) + \int_{t_0}^{t} z_2(s) y(s) ds, \qquad \forall t \in [t_0, T].$$

Show that

$$y(t) \leq z_1(t) + \int_{t_0}^{t} z_1(s) z_2(s) e^{\int_s^t z_2(r) dr} ds, \qquad \forall t \in [t_0, T].$$

In the case where z_1 *is non-decreasing, show that*

$$y(t) \leq z_1(t) e^{\int_a^t z_2(s) ds}.$$

Exercise 7. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a Lipschitz function (in all of* $\mathbb{R}$*). Show that the solution of the IVP:*

$$y' = f(y), \qquad y(t_0) = y_0$$

is global, ie. it is defined in the whole real line $\mathbb{R}$.

[Hint: Use the continuation of solutions theorem.]